

Conformal Spinning Quantum Particles in Complex Minkowski Space as Constrained Nonlinear Sigma Models in $U(2, 2)$ and Born's Reciprocity

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Abstract

We revise the use of 8-dimensional conformal, complex (Cartan) domains as a base for the construction of conformally invariant quantum (field) theory, either as phase or configuration spaces. We follow a gauge-invariant Lagrangian approach (of nonlinear sigma-model type) and use a generalized Dirac method for the quantization of constrained systems, which resembles in some aspects the standard approach to quantizing coadjoint orbits of a group G . Physical wave functions, Haar measures, orthonormal basis and reproducing (Bergman) kernels are explicitly calculated in and holomorphic picture in these Cartan domains for both scalar and spinning quantum particles. Similarities and differences with other results in the literature are also discussed and an extension of Schwinger's Master Theorem is commented in connection with closure relations. An adaptation of the Born's Reciprocity Principle (BRP) to the conformal relativity, the replacement of space-time by the 8-dimensional conformal domain at short distances and the existence of a maximal acceleration are also put forward.

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1 Introduction

Complex manifolds and, in particular, Cartan classical domains have been studied for many years by mathematicians and theoretical physicists (see e.g. [1] and references therein for a review). In this article we are interested in the Lie ball

$$\mathbb{D} = SO(4, 2)/(SO(4) \times SO(2)) = SU(2, 2)/S(U(2) \times U(2)),$$

which can be mapped one-to-one onto the 8-dimensional forward/future tube domain

$$\mathbb{T} = \{x^\mu + iy^\mu \in \mathbb{C}^{1,3}, \quad y^0 > \|\vec{y}\|\}$$

of the complex Minkowski space $\mathbb{C}^{1,3}$ through a Cayley transformation (see next Section for more details). Both manifolds can be considered as the phase space of massive conformal particles and there is a renewed interest in its quantization (see e.g. [2] and references therein for a survey). The presentation followed in the literature is of geometric (twistor [3, 4] and Konstant-Kirillov-Souriau [5, 6] descriptions) and representation-theoretic [7, 8] nature. Here we shall adopt a (sigma-model-type) Lagrangian approach to the subject and we shall use a generalized Dirac method for the quantization of constrained systems which resembles in some aspects the particular approach to quantizing coadjoint orbits of a group G developed many years ago in [9] (see also [10] and [11] for interesting examples in $G = SU(3)$).

We share with many authors (namely, [1, 2, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]) the belief that the use of complex Minkowski 8-dimensional space as a base for the construction of quantum (field) theory is not only useful from the technical point of view but can be of great physical importance. Actually, as suggested in [14], the conformal domain \mathbb{D} could be considered as the replacement of the space-time at short distances (at the “microscale”). This interpretation is based on Born’s Reciprocity Principle (BRP) [15, 16], originally intended to merging quantum theory and general relativity. The reciprocity symmetry between coordinates x_μ and momenta p_μ states that the laws of nature are (or should be) invariant under the transformations

$$(x_\mu, p_\mu) \rightarrow (\pm p_\mu, \mp x_\mu). \tag{1}$$

The word “reciprocity” is used in analogy with the lattice theory of crystals, where some physical phenomena (like the theory of diffraction) are sometimes better described in the p -space by means of the reciprocal (Bravais) lattice. The argument here is that Born’s reciprocity implies that there must be a reciprocally conjugate relativity principle according to which the rate of change of momentum (force) should be bounded by a universal constant b , much in the same way the usual relativity principle implies a bound of the rate of change of position (velocity) by the speed of light c . As a consequence of the BRP, there must exist a minimum (namely, Planck) length $\ell_{\min} = \sqrt{\hbar c/b}$.

This symmetry led Born to conjecture that the basic underlying physical space is the 8-dimensional $\{x_\mu, p_\mu\}$ and to replace the Poincaré invariant line element $d\tau^2 = dx_\mu dx^\mu$

by the Finslerian-like metric (see [17, 18] for an extension to Born-Clifford phase spaces)

$$d\tilde{\tau}^2 = dx_\mu dx^\mu + \frac{\ell_{\min}^4}{\hbar^2} dp_\mu dp^\mu. \quad (2)$$

From the BRP point of view, local (versus extended) field theories like Klein-Gordon's represent the “point-particle limit” $\ell_{\min} \rightarrow 0$, for which the reciprocal symmetry is broken. Also, the Minkowski spacetime is interpreted either as a local ($\ell_{\min} \rightarrow 0$) version or as a high-energy-momentum-transfer limit ($b \rightarrow \infty$) of this 8-dimensional phase-space domain. Moreover, putting $dp_\mu/d\tau = md^2x_\mu/d\tau^2 = ma_\mu$, with $m = b\ell_{\min}/c^2$ a (namely, Planck) mass and a_μ the proper acceleration (with $a^2 \leq 0$, space-like), one can write the previous extended line element as

$$d\tilde{\tau} = d\tau \sqrt{1 - \frac{|a^2|}{a_{\max}^2}}, \quad (3)$$

which naturally leads to a *maximal (proper) acceleration* $a_{\max} = c^2/\ell_{\min}$. The existence and physical consequences of a maximal acceleration was already derived by Caianiello [19]. Many papers have been published in the last years (see e.g. [20] and references therein), each one introducing the maximal acceleration starting from different motivations and from different theoretical schemes. Among the large list of physical applications of Caianiello's model we would like to point out the one in cosmology which avoids an initial singularity while preserving inflation. Also, a maximal-acceleration relativity principle leads to a variable fine structure “constant” α [20], according to which α could have been extremely small (zero) in the early Universe and then all matter in the Universe could have emerged via the Fulling-Davies-Unruh-Hawking effect (vacuum radiation due to the acceleration with respect to the vacuum frame of reference) [23, 24, 25, 26].

There has been group-theoretical revisions of the BRP like [21, 22] replacing the Poincaré by the Canonical (or Quaplectic) group of reciprocal relativity, which enjoys a richer structure than Poincaré. In this article we pursue a different reformulation of BRP as a natural symmetry inside the conformal group $SO(4, 2)$ and the replacement of space-time by the 8-dimensional conformal domain \mathbb{D} or \mathbb{T} at short distances. We believe that new interesting physical phenomena remain to be unravelled inside this framework. Actually, in a coming paper [27] (see also [28] for a previous related work), we shall discuss a group-theoretical revision of the Unruh effect [25] as a spontaneous breakdown of the conformal symmetry and the consequences of a maximal acceleration. Also, a wavelet transform on the tube domain \mathbb{T} , based on the conformal group, could provide a way to analyze wave packets localized in both: space and time. Important developments in this direction have been done in [29, 30] for electromagnetic (massless) signals and [31] for fields with continuous mass spectrum.

In this article we shall study the geometrical and quantum mechanical underlying framework. We shall follow a gauge-invariant (singular) Lagrangian approach of nonlinear sigma-model type and we shall use a generalized Dirac method for the quantization of constrained systems.

The paper is organized as follows. In Section 2 we briefly review the conformal group $SO(4, 2) \simeq SU(2, 2)$, its Lie algebra generators and commutators, and provide different

coordinate systems for the conformal domains \mathbb{D} and \mathbb{T} ; in this Section we also introduce the concept of BRP in a conformally invariant setting. Section 3 is devoted to the Lagrangian formulation of conformally invariant nonlinear sigma-models on the conformal domains (either as configuration or phase spaces) and the study of their gauge invariance. The quantization of these models (for the case of Lagrangians linear in velocities) is accomplished in Section 4 by using a generalized Dirac method for the quantization of constrained systems which resembles in some aspects the particular approach to quantizing coadjoint orbits of G . Physical wave functions, Haar measures, orthonormal basis and reproducing (Bergman) kernels are explicitly calculated in an holomorphic picture in the Cartan domain \mathbb{D} , for both scalar and spinning quantum particles in subsections 4.1 and 4.2, respectively. Similarities and differences with other results in the literature are also discussed and an extension of the Schwinger Master Theorem is commented in connection with closure relations. In Section 5 we translate (through an equivariant map) all the constructions above to the tube domain \mathbb{T} , where we enjoy more physical intuition. We comment on Kähler structures and generalized Born-like line elements and the existence of a maximal acceleration for conformal (quantum) particles. The last Section 6 is devoted to comments and outlook where we point out an interesting connection between BRP and CPT symmetry inside the conformal group and discuss on the appearance of a maximal acceleration in this scheme.

2 The conformal symmetry in 1+3D: coordinate systems and generators

The conformal group $SO(4, 2)$ is comprised of Poincaré (spacetime translations $b^\mu \in \mathbb{R}^{1,3}$ and Lorentz $\Lambda_\nu^\mu \in SO(3, 1)$) transformations augmented by dilations ($\rho = e^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations, SCT, $a^\mu \in \mathbb{R}^{1,3}$) which, in Minkowski spacetime, have the following realization:

$$\begin{aligned} x'^\mu &= x^\mu + b^\mu, & x'^\mu &= \Lambda_\nu^\mu(\omega)x^\nu, \\ x'^\mu &= \rho x^\mu, & x'^\mu &= \frac{x^\mu + a^\mu x^2}{1 + 2ax + a^2 x^2}, \end{aligned} \quad (4)$$

respectively. The interpretation of SCT as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by (see e.g., [32, 33, 34]), although alternative meanings have also been proposed. One is related to the Weyl's idea of different lengths in different points of space time [35]: “the rule for measuring distances changes at different positions”. Other is Kastrup's interpretation of SCT as geometrical gauge transformations of the Minkowski space [36] (for this point see later on Eq. (40)).

The generators of the transformations (4) are easily deduced:

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial x^\mu}, & M_{\mu\nu} &= x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \\ D &= x^\mu \frac{\partial}{\partial x^\mu}, & K_\mu &= -2x_\mu x^\nu \frac{\partial}{\partial x^\nu} + x^2 \frac{\partial}{\partial x^\mu} \end{aligned} \quad (5)$$

and they close into the conformal Lie algebra

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}, \\
[P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho, \quad [P_\mu, P_\nu] = 0, \\
[K_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho, \quad [K_\mu, K_\nu] = 0, \\
[D, P_\mu] &= -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [D, M_{\mu\nu}] = 0, \\
[K_\mu, P_\nu] &= 2(\eta_{\mu\nu}D + M_{\mu\nu}).
\end{aligned} \tag{6}$$

We shall argue later that P_μ and K_μ are conjugated variables (they can not be simultaneously measured) and that D can be taken to be the generator of (proper) time translations (i.e., the Hamiltonian). A BRP-like symmetry manifests here in the form:

$$P_\mu \rightarrow K_\mu, \quad K_\mu \rightarrow P_\mu, \quad D \rightarrow -D, \tag{7}$$

which leaves the commutation relations (6) unaltered. This symmetry can also be seen in the quadratic Casimir operator:

$$C_2 = D^2 - \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu) = D^2 - \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + P_\mu K^\mu + 4D, \tag{8}$$

which generalizes the Poincaré Casimir $P^2 = P_\mu P^\mu$, just as $d\tilde{\tau}$ in (2) generalizes the Poincaré invariant line element $d\tau$. We shall provide a conformal invariant line element similar to $d\tilde{\tau}$ later in Section 5.

Any group element $g \in SO(4, 2)$ (near the identity element 1) could be written as the exponential map

$$g = \exp(u), \quad u = \tau D + b^\mu P_\mu + a^\mu K_\mu + \omega^{\mu\nu} M_{\mu\nu}, \tag{9}$$

of the Lie-algebra element u (see [37, 38]). The compactified Minkowski space $\mathbb{M} = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1 \simeq U(2)$ can be obtained as the coset $\mathbb{M} = SO(4, 2)/\mathbb{W}$, where \mathbb{W} denotes the Weyl subgroup generated by $K_\mu, M_{\mu\nu}$ and D (i.e., a Poincaré subgroup $\mathbb{P} = SO(3, 1) \otimes \mathbb{R}^4$ augmented by dilations \mathbb{R}^+). The Weyl group \mathbb{W} is the stability subgroup (the little group in physical usage) of $x^\mu = 0$.

There is another interesting realization of the conformal Lie algebra (6) in terms of gamma matrices in, for instance, the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix},$$

where $\check{\sigma}^\mu \equiv \sigma_\mu$ (we are using the convention $\eta = \text{diag}(1, -1, -1, -1)$ for the Minkowski metric) and σ^μ are the Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, the choice

$$\begin{aligned}
D &= \frac{\gamma^5}{2}, \quad M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4} \begin{pmatrix} \sigma^\mu \check{\sigma}^\nu - \sigma^\nu \check{\sigma}^\mu & 0 \\ 0 & \check{\sigma}^\mu \sigma^\nu - \check{\sigma}^\nu \sigma^\mu \end{pmatrix}, \\
P^\mu &= \gamma^\mu \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \quad K^\mu = \gamma^\mu \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \check{\sigma}^\mu & 0 \end{pmatrix}
\end{aligned} \tag{10}$$

fulfils the commutation relations (6). These are the Lie algebra generators of the fundamental representation of the four cover of $SO(4, 2)$:

$$SU(2, 2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}_{4 \times 4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \right\}, \quad (11)$$

with Γ a 4×4 hermitian form of signature $(++--)$. In particular, taking $\Gamma = \gamma^5$, the 2×2 complex matrices A, B, C, D in (11) satisfy the following restrictions:

$$g^{-1}g = I_{4 \times 4} \Leftrightarrow \begin{cases} D^\dagger D - B^\dagger B = \sigma^0 \\ A^\dagger A - C^\dagger C = \sigma^0 \\ A^\dagger B - C^\dagger D = 0, \end{cases} \quad (12)$$

together with those of $gg^{-1} = I_{4 \times 4}$. In this article we shall work with $G = U(2, 2)$ instead of $SO(4, 2)$ and we shall use a set of complex coordinates to parametrize G . This parametrization will be adapted to the non-compact complex Grassmannian $\mathbb{D} = G/H$ of the maximal compact subgroup $H = U(2)^2$. It can be obtained through a block-orthonormalization process with metric $\Gamma = \gamma^5$ of the matrix columns of:

$$\begin{pmatrix} \sigma^0 & 0 \\ Z^\dagger & \sigma^0 \end{pmatrix} \rightarrow g = \begin{pmatrix} \sigma^0 & Z \\ Z^\dagger & \sigma^0 \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \begin{cases} \Delta_1 = (\sigma^0 - ZZ^\dagger)^{-1/2} \\ \Delta_2 = (\sigma^0 - Z^\dagger Z)^{-1/2} \end{cases}.$$

Actually, we can identify

$$Z = Z(g) = BD^{-1}, Z^\dagger = Z^\dagger(g) = CA^{-1}, \Delta_1 = (AA^\dagger)^{1/2}, \Delta_2 = (DD^\dagger)^{1/2}. \quad (13)$$

From (12), we obtain the positive-matrix conditions $AA^\dagger > 0$ and $DD^\dagger > 0$, which are equivalent to:

$$\sigma^0 - ZZ^\dagger > 0, \sigma^0 - Z^\dagger Z > 0. \quad (14)$$

Moreover, from the top condition of (12), we arrive at the determinant restriction:

$$\det(ZZ^\dagger) = \det(B^\dagger B) \det(\sigma^0 + B^\dagger B)^{-1} < 1, \quad (15)$$

which, together with $\det(\sigma^0 - ZZ^\dagger) = 1 - \text{tr}(ZZ^\dagger) + \det(ZZ^\dagger) > 0$, implies that $\text{tr}(ZZ^\dagger) < 2$. Thus, we can identify the symmetric complex Cartan domain

$$\mathbb{D} = G/H = \{Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : \sigma^0 - ZZ^\dagger > 0\} \quad (16)$$

with an open subset of the eight-dimensional ball with radius $\sqrt{2}$. Moreover, the compactified Minkowski space \mathbb{M} is the Shilov boundary $U(2) = \{Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : Z^\dagger Z = ZZ^\dagger = \sigma^0\}$ of \mathbb{D} .

There is a one-to-one mapping from \mathbb{D} onto the future tube domain

$$\mathbb{T} = \{W = X + iY \in \text{Mat}_{2 \times 2}(\mathbb{C}) : Y > 0\}, \quad (17)$$

of the complex Minkowski space $\mathbb{C}^{1,3}$, with $X = x_\mu \sigma^\mu$ and $Y = y_\mu \sigma^\mu$ hermitian matrices and $Y > 0 \Leftrightarrow y^0 > \|\vec{y}\|$. This map is given by the Cayley transformation and its inverse:

$$Z \rightarrow W(Z) = i(\sigma^0 - Z)(\sigma^0 + Z)^{-1}, \quad W \rightarrow Z(W) = (\sigma^0 - iW)^{-1}(\sigma^0 + iW). \quad (18)$$

This is the 3+1-dimensional analogue of the usual map from the unit disk onto the upper half-plane in two dimensions. Actually, the forward tube domain \mathbb{T} is naturally homeomorphic to the quotient G/H in a new realization of G in terms of matrices f which preserve $\Gamma = \gamma^0$, instead of $\Gamma = \gamma^5$; that is, $f^\dagger \gamma^0 f = \gamma^0$. Both realizations of G are related by the map

$$g \rightarrow f = \Upsilon g \Upsilon^{-1}, \quad \Upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma^0 & -\sigma^0 \\ \sigma^0 & \sigma^0 \end{pmatrix}. \quad (19)$$

We shall come again to this “forward tube domain” realization later on Section 5.

Let us proceed by giving a complete local parametrization of G adapted to the fibration $H \rightarrow G \rightarrow \mathbb{D}$. Any element $g \in G$ (in the present patch, containing the identity element) admits the Iwasawa decomposition

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Delta_1 & Z\Delta_2 \\ Z^\dagger \Delta_1 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad (20)$$

where the last factor

$$U_1 = \Delta_1^{-1}A, U_2 = \Delta_2^{-1}D$$

belongs to H ; i.e., $U_1, U_2 \in U(2)$. Likewise, a parametrization of any $U \in U(2)$ (in a patch containing the identity), adapted to the quotient $\mathbb{S}^2 = U(2)/U(1)^2$, is (the Hopf fibration)

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & z\delta \\ -\bar{z}\delta & \delta \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \quad (21)$$

where $z = b/d \in \overline{\mathbb{C}} \simeq \mathbb{S}^2$ (the one-point compactification of \mathbb{C} by inverse stereographic projection), $\delta = (1 + z\bar{z})^{-1/2}$ and $e^{i\alpha} = a/|a|, e^{i\beta} = d/|d|$.

Sometimes it will be more convenient for us to use the following compact notation for the sixteen coordinates of $U(2, 2)$:

$$\left\{ \begin{array}{cccc} \alpha_1 & z_1 & Z_{11} & Z_{12} \\ -\bar{z}_1 & \beta_1 & Z_{21} & Z_{22} \\ \bar{Z}_{11} & \bar{Z}_{21} & \alpha_2 & z_2 \\ \bar{Z}_{12} & \bar{Z}_{22} & -\bar{z}_2 & \beta_2 \end{array} \right\} = \left\{ \begin{array}{cccc} x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{array} \right\} = \{x_\beta^\alpha(g)\}, \quad (22)$$

The set of coordinates $\{x_\beta^\alpha\}$ is adapted to the new Lie algebra basis of step operator matrices $(X_\alpha^\beta)_\mu^\nu \equiv \delta_\alpha^\nu \delta_\mu^\beta$ fulfilling the commutation relations:

$$[X_{\alpha_1}^{\beta_1}, X_{\alpha_2}^{\beta_2}] = \delta_{\alpha_1}^{\beta_2} X_{\alpha_2}^{\beta_1} - \delta_{\alpha_2}^{\beta_1} X_{\alpha_1}^{\beta_2}, \quad (23)$$

and the usual orthogonality properties:

$$\text{tr}(X_\alpha^\beta X_\gamma^\rho) = \delta_\alpha^\rho \delta_\gamma^\beta.$$

The Cartan (maximal Abelian) subalgebra $u(1)^4 \subset \mathcal{G}$ is made of diagonal operators $\{X_\alpha^\alpha, \alpha = 1, \dots, 4\}$.

Another realization of the conformal Lie algebra that will be useful for us is the one given in terms of left- and right-invariant vector fields, as generators of right- and left-translations of G ,

$$[\mathcal{U}_g^R \psi](g') = \psi(g'g), \quad [\mathcal{U}_g^L \psi](g') = \psi(g^{-1}g'), \quad (24)$$

on complex functions $\psi : G \rightarrow \mathbb{C}$, respectively. Denoting by

$$\theta^L = -ig^{-1}dg = \theta_\beta^\alpha X_\alpha^\beta = \theta_{\beta\nu}^{\alpha\mu} dx_\mu^\nu X_\alpha^\beta \quad (25)$$

the left-invariant Maurer-Cartan 1-form, the left-invariant vector fields L_α^β are defined by duality $\theta_\beta^\alpha(L_\rho^\sigma) = \delta_\rho^\sigma \delta_\beta^\alpha$. The same applies to right-invariant 1-forms $\theta^R = -idgg^{-1}$ in relation with right-invariant vector fields R_α^β . They can also be computed through the group law $g'' = g'g$ as:

$$L_\alpha^\beta(g) \equiv \left. \frac{\partial x_\nu^\mu(gg')}{\partial x_\beta^\alpha(g')} \right|_{g'=1} \frac{\partial}{\partial x_\mu^\nu(g)}, \quad R_\alpha^\beta(g) \equiv \left. \frac{\partial x_\nu^\mu(g'g)}{\partial x_\beta^\alpha(g')} \right|_{g'=1} \frac{\partial}{\partial x_\mu^\nu(g)}. \quad (26)$$

The quadratic Casimir operator (8) now adopts the compact form:

$$C_2 = L_\alpha^\beta L_\beta^\alpha = R_\alpha^\beta R_\beta^\alpha.$$

Both sets of vector fields will be essential in our quantization procedure, the first ones (L) as generators of gauge transformations and the second ones (R) as the symmetry operators of our theory.

3 Non-linear sigma models on G

The actual Lagrangian for quantum mechanical geodesic free motion on G , as a configuration space, is given by:

$$\mathcal{L}_G(g, \dot{g}) = \frac{1}{2} \text{tr}(\vartheta^L)^2 = \frac{1}{2} \vartheta_\beta^\alpha \vartheta_\alpha^\beta = \frac{1}{2} g_{\mu\rho}^{\nu\sigma}(x) \dot{x}_\nu^\mu \dot{x}_\sigma^\rho, \quad (27)$$

where we are denoting by

$$\vartheta^L = -ig^{-1}\dot{g} = \vartheta_\beta^\alpha X_\alpha^\beta = \vartheta_{\beta\mu}^{\alpha\nu} \dot{x}_\nu^\mu X_\alpha^\beta$$

the restriction of (25) to trajectories $g = g(t)$ and writing the natural metric on G , $g_{\mu\rho}^{\nu\sigma} = \vartheta_{\beta\mu}^{\alpha\nu} \vartheta_{\alpha\rho}^{\beta\sigma}$, in terms of vielbeins ϑ_β^α . The equations of motion derived from (27) are: $\dot{\vartheta}^L = 0$, which can be converted into the standard form of geodesic motion

$$\ddot{x}^a + \Gamma_{bc}^a(x) \dot{x}^b \dot{x}^c = 0$$

by introducing the Levi-Civita connection Γ_{bc}^a [here we used an alternative indexation $a = (\alpha\beta) = 1, \dots, 16$, to simplify expressions]. The phase space of this theory is the cotangent bundle T^*G , which can be identified with the product of G and its Lie algebra \mathcal{G} in a suitable way.

It can be shown that the Lagrangian (27) is G -invariant under both: left- and right-rigid transformations, $g(t) \rightarrow g'g(t)$ and $g(t) \rightarrow g(t)g'$, respectively; that is, \mathcal{L}_G is chiral. This chirality is partially broken when we reduce the dynamics from G to certain cosets G/G^0 , with G^0 the isotropy subgroup of a given Lie algebra element of the form

$$X_0 = \sum_{\alpha=1}^4 \lambda_\alpha X_\alpha \quad (28)$$

(with λ_α some real constants) under the adjoint action $X_0 \rightarrow gX_0g^{-1}$ of G on its Lie algebra \mathcal{G} . Actually, the new Lagrangian on G/G^0 can be written as a “partial trace”:

$$\mathcal{L}_{G/G^0}(g, \dot{g}) = \frac{1}{2} \text{tr}_{G/G^0}(\vartheta^L)^2 \equiv \frac{1}{2} \text{tr}([X_0, \vartheta^L])^2 = \frac{1}{2} \sum_{\alpha, \beta=1}^N (\lambda_\alpha - \lambda_\beta)^2 \vartheta_\beta^\alpha \vartheta_\alpha^\beta. \quad (29)$$

For example, choosing $X_0 = \frac{\lambda}{2} \gamma^5 = \lambda D$ (the dilation) we have $G^0 = H = U(2)^2$ (the maximal compact subgroup) and G/G^0 the eight-dimensional domain \mathbb{D} . For $\lambda_\alpha \neq \lambda_\beta, \forall \alpha, \beta = 1, \dots, 4$, the isotropy subgroup of X_0 is the maximal Abelian subgroup $G^0 = U(1)^4$ and $G/G^0 = \mathbb{F}$ is a twelve-dimensional “pseudo-flag” (non-compact) manifold. It is obvious that \mathcal{L}_{G/G^0} is still invariant under general rigid left-transformations $g(t) \rightarrow g'g(t)$. However, this Lagrangian is now singular or, equivalently:

Proposition 3.1. *The Lagrangian (29) is gauge invariant under local right-transformations*

$$g(t) \rightarrow g(t)g_0(t), \quad \forall g_0(t) \in G^0 \quad (30)$$

Proof: we have that:

$$\vartheta^L = -ig^{-1}\dot{g} \rightarrow \vartheta'^L = -ig_0^{-1}g^{-1}(\dot{g}g_0 + g\dot{g}_0) = g_0^{-1}\vartheta^L g_0 - ig_0^{-1}\dot{g}_0$$

and

$$[X_0, \vartheta'^L] = g_0^{-1}[X_0, \vartheta^L]g_0,$$

since G^0 is the isotropy subgroup of X_0 , which means $[X_0, g_0] = 0 = [X_0, \dot{g}_0]$. The cyclic property of the trace completes the proof ■

We have considered so far G/G^0 as a configuration space. In this article, we shall be rather interested in G/G^0 as a phase space. For example, we shall consider \mathbb{D} [or the tube domain (17) of the complex Minkowski space $\mathbb{C}^{1,3}$] as a (complex) phase space of four-position x^μ and four-momenta y^ν , in itself. This situation will require a new singular Lagrangian of the form:

$$\mathcal{L}(g, \dot{g}) = \text{tr}(X_0 \vartheta^L) = \sum_{\alpha=1}^4 \lambda_\alpha \vartheta_\alpha^\alpha. \quad (31)$$

Again, this Lagrangian is left- G -invariant under rigid transformations. The difference now is that it is linear in velocities \dot{x} . Moreover, we shall prove that:

Proposition 3.2. *The Lagrangian (31) is gauge (semi-)invariant under local right- transformations*

$$g(t) \rightarrow g(t)g_0(t), \quad \forall g_0(t) \in G^0 \quad (32)$$

up to a total time derivative, i.e.,

$$\mathcal{L} \rightarrow \mathcal{L} + \Delta\mathcal{L}, \quad \Delta\mathcal{L} = -i\text{tr}(X_0 g_0^{-1} \dot{g}_0) = \frac{d\tau}{dt}, \quad \tau = \sum_{\alpha=1}^4 \lambda_\alpha x_\alpha^\alpha. \quad (33)$$

Proof: We shall just consider the two important cases for us:

1. $\lambda_\alpha \neq \lambda_\beta, \forall \alpha \neq \beta \Rightarrow G^0 = U(1)^4, G/G^0 = \mathbb{F}$
2. $X_0 = \lambda D = \frac{\lambda}{2} \gamma^5 \Rightarrow G^0 = H = U(2)^2, G/G^0 = \mathbb{D}.$

For the first case, any $g_0 \in G^0$ can be written as $g_0 = \exp(ix_\alpha^\alpha X_\alpha^\alpha)$ and $\dot{g}_0 = ig_0 \dot{x}_\alpha^\alpha X_\alpha^\alpha$ because G^0 is Abelian; therefore

$$\Delta\mathcal{L} = -i\text{tr}(X_0 g_0^{-1} \dot{g}_0) = \sum_{\beta=1}^4 \lambda_\beta \dot{x}_\alpha^\alpha \text{tr}(X_\beta^\beta X_\alpha^\alpha) = \sum_{\alpha=1}^4 \lambda_\alpha \dot{x}_\alpha^\alpha.$$

For the second case, $g_0 = \exp(i\varphi I + i\tau' D + i\omega^{\mu\nu} M_{\mu\nu}) = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in H$. Disregarding the trivial global phase φ , it is clear that for dilations $g_0 = d_0 = e^{i\tau' D}$ we have $\dot{d}_0 = i\dot{\tau}' D e^{i\tau' D}$ and

$$-i\text{tr}(X_0 d_0^{-1} \dot{d}_0) = \lambda \dot{\tau}' \text{tr}(D^2) = \lambda \dot{\tau}' = \dot{\tau},$$

where $\tau \equiv \lambda\tau'$. For Lorentz transformations $g_0 = m_0 = \exp(i\omega^{\mu\nu} M_{\mu\nu})$ we have $\Delta\mathcal{L} = 0$ since $\text{tr}(DM_{\mu\nu}) = 0$, which is a direct consequence of the orthogonality properties of the Pauli matrices $\text{tr}(\sigma^\mu \sigma^\nu) = 2\delta^{\mu\nu}$. ■

Remark 3.3. We can always fix the gauge to $\tau(t) = t$. In the case $X_0 = \lambda D$, this implies that the dilation operator D will play the role of the Hamiltonian of the quantum theory. The replacement of time translations by dilations as dynamical equations of motion has been considered in [39] and in [40] when quantizing field theories on space-like Lorentz-invariant hypersurfaces $x^2 = x^\mu x_\mu = \tau^2 = \text{constant}$. In other words, if one wishes to proceed from one surface at $x^2 = \tau_1^2$ to another at $x^2 = \tau_2^2$, this is done by scale transformations; that is, D is the evolution operator in a proper time τ □.

4 Quantum mechanics in the phase space G/G^0

We shall see that the constants λ_α label the (lowest weight) irreducible representations of G on which the Hilbert space of our theory is constructed. There are several ways of seeing that the values of λ_α are quantized. One way is through the path integral method. To examine this explicitly, consider the transition amplitude from an initial point g_1 at

$t = t_1$ to a final point g_2 at $t = t_2$. For each path $g(t)$ connecting g_1 and g_2 , there are many gauge equivalent paths

$$g'(t) = g(t)g_0(t), \quad g_0(t) \in G^0, \quad g_0(t_1) = g_0(t_2) = 1$$

that must contribute to the sum of the path integral with the same amplitude, that is:

$$e^{i \int_{t_1}^{t_2} dt \mathcal{L}(g, \dot{g})} = e^{i \int_{t_1}^{t_2} dt \mathcal{L}(g', \dot{g}')} = e^{i \int_{t_1}^{t_2} dt \mathcal{L}(g, \dot{g})} e^{i \int_{t_1}^{t_2} dt \Delta \mathcal{L}(g, \dot{g})} \Rightarrow e^{i \int_{t_1}^{t_2} dt \Delta \mathcal{L}(g, \dot{g})} = 1.$$

Using (33), the last expression can be written as $\exp(i(\tau(t_2) - \tau(t_1))) = 1$ which, together with the fact that

$$g_0(t_{1,2}) = e^{i \sum_{\alpha} x_{\alpha}^{\alpha}(t_{1,2})} = 1 \Leftrightarrow x_{\alpha}^{\alpha}(t_{1,2}) = 2\pi n_{1,2}^{\alpha}, \quad n_{1,2}^{\alpha} \in \mathbb{Z},$$

means that λ_{α} must be an integer number. Considering coverings of G , one can relax the integer to a half-integer condition, as happens with $SU(2)$ in relation with $SO(3)$.

Other alternative way to the path-integral description of realizing the integrality of λ_{α} is through the following operator (representation-theoretic) description. At the quantum level, finite-right gauge transformations like (32) induce constraints on “physical” wave functions $\psi(g)$ as:

$$\psi(gg_0) = \mathcal{U}_0^{\lambda}(g_0)\psi(g), \quad g_0 \in G^0 \quad (34)$$

where we are allowing ψ to transform non-trivially according to a representation \mathcal{U}_0^{λ} of G^0 of index λ . This could be seen as a generalization of the original Dirac approach to the quantization of constrained systems (where \mathcal{U}_0^{λ} is taken to be trivial) which allows new inequivalent quantizations labelled by λ_{α} (see e.g. [41, 42, 43, 44] for several approaches to the subject). The finite constraint condition (34) can be written in infinitesimal form as

$$L_{\alpha}^{\alpha}\psi = \lambda_{\alpha}\psi, \quad \alpha = 1, \dots, 4, \quad (35)$$

where we have used the fact that left-invariant vector fields (26) are generators of finite right-transformations. In the parametrization $\{x_{\beta}^{\alpha}\}$, the left-invariant vector fields L_{α}^{β} fulfill the same commutation relations as the step operator matrices (23). Therefore, when acting on physical/constrained states (35), they satisfy creation and annihilation harmonic-oscillator-like commutation relations:

$$[L_{\alpha}^{\beta}, L_{\beta}^{\alpha}] = (\lambda_{\beta} - \lambda_{\alpha}) \quad (\text{no sum on } \alpha, \beta).$$

We shall work in a holomorphic picture, which means that constrained wave functions (35) will be further restricted by holomorphicity conditions:

$$L_{\alpha}^{\beta}\psi = 0, \quad \forall \alpha > \beta = 1, 2, 3. \quad (36)$$

In fact, looking at (26), for $g \in G$ near the identity we have $L_{\alpha}^{\beta}(g) \sim \partial/\partial x_{\beta}^{\alpha}$ so that $L_{\alpha}^{\beta}\psi = 0$ means, roughly speaking, that $\psi(g)$ does not depend on the variables x_{β}^{α} , $\alpha > \beta = 1, 2, 3$ in (22), that is, ψ is holomorphic. The complementary option $L_{\alpha}^{\beta}\psi = 0, \forall \beta > \alpha = 1, 2, 3$

then leads to anti-holomorphic functions. Those readers familiar with Geometric Quantization [5, 45] will identify the constraint equations (35) and (36) as *polarization* conditions (see also [46] for a Group Approach to Quantization scheme and [47] for the extension of first-order polarizations to higher-order polarizations), intended to reduce the left-representation \mathcal{U}^L (24) of G , on complex wave functions ψ , to G/G^0 . Also, the constraints (35) and (36) are exactly the defining relations of a lowest-weight representation.

4.1 Conformal scalar quantum particles

Firstly we shall consider the (spin-less) case $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 \equiv -\lambda/2$, that is, $X_0 = \frac{\lambda}{2}\gamma^5 = \lambda D$, and we shall call λ the *conformal, scale or mass dimension*. In this case the gauge group is the maximal compact subgroup $G^0 = H = U(2)^2$ and the phase space is the eight-dimensional domain $\mathbb{D} = G/G^0$.

4.1.1 Constraint conditions and physical wave functions

The constraint conditions (35) can now be enlarged to

$$D^L\psi = -\frac{1}{2}(L_1^1 + L_2^2 - L_3^3 - L_4^4)\psi = \lambda\psi, \quad M_{\mu\nu}^L\psi = 0. \quad (37)$$

which renders translation (P_μ) and acceleration (K_ν) generators into conjugated variables. In fact, the last commutator of (6), on constrained (physical) wave functions (37), gives:

$$[K_\mu^L, P_\nu^L]\psi = 2\lambda\eta_{\mu\nu}\psi, \quad (38)$$

which states that K_μ and P_μ can not be simultaneously measured, the conformal dimension λ playing here the role of the Planck constant \hbar . Note that K_μ and P_μ are conjugated but not *canonically* conjugated as such. We address the reader to Refs. [48, 49] for other definitions of quantum observables associated with positions in space-time, namely

$$X_\mu = M_{\nu\mu} \cdot \frac{P^\nu}{P^2} + D \cdot \frac{P_\mu}{P^2} \quad (39)$$

(dot means symmetrization), fulfilling canonical commutation relations $[X_\mu, P_\nu] = \eta_{\mu\nu}$ inside the conformal (enveloping) algebra (6).

A further restriction

$$K_\mu^L\psi = 0 \quad (40)$$

selects the holomorphic (“position”) representation. Indeed, let us prove that:

Theorem 4.1. *The general solution to (37) and (40) can be factorized as:*

$$\psi_\lambda(g) = \mathcal{W}_\lambda(g)\phi(Z), \quad (41)$$

where the “ground state”

$$\begin{aligned} \mathcal{W}_\lambda(g) &= \det(D)^{-\lambda} = \det(\sigma^0 - Z^\dagger Z)^{\lambda/2} \det(U_2)^{-\lambda} \\ &= (1 - \text{tr}(Z^\dagger Z) + \det(Z^\dagger Z))^{\lambda/2} \det(U_2)^{-\lambda} \end{aligned} \quad (42)$$

is a particular solution of (37,40) and ϕ is the general solution for the trivial representation $\lambda = 0$ of $G^0 = H$ (actually, an arbitrary, analytic holomorphic function of Z), for the decomposition (20) of an element $g \in G$.

Proof: A generic proof (also valid for other symmetry groups) that the general solution of (37,40) admits a factorization of the form (41) can be found in the Proposition 3.3 of [50]. Here we shall just prove that (41) is a solution of (37,40). Indeed, by applying a finite right translation (24) on $\mathcal{W}_\lambda(g)$:

$$\begin{aligned} [\mathcal{U}_g^R \mathcal{W}_\lambda](g) &= \mathcal{W}_\lambda(gg') = \det(D'')^{-\lambda} = \det(CB' + DD')^{-\lambda} \\ &= \det(D')^{-\lambda} \det(CZ' + D)^{-\lambda}, \end{aligned} \quad (43)$$

we see that $\mathcal{W}_\lambda(gg')$ is not affected by translations by $Z'^\dagger = Z^\dagger(g') = C'A'^{-1}$. Infinitesimally, it means that $K_\mu^L \mathcal{W}_\lambda(g) = 0$, according to the lower-triangular choice of the generator K_μ in (10). For Lorentz transformations we have $B' = 0 = C'$ and $\det(A') = 1 = \det(D')$ and therefore $\mathcal{W}_\lambda(gg') = \mathcal{W}_\lambda(g)$, that is $M_{\mu\nu}^L \mathcal{W}_\lambda(g) = 0$. For dilations we have $B' = 0 = C'$ and $A' = e^{i\tau/2} \sigma^0 = D'^\dagger$, which gives $\mathcal{W}_\lambda(gg') = e^{i\lambda\tau} \mathcal{W}_\lambda(g)$ or $D^L \mathcal{W}_\lambda(g) = \lambda \mathcal{W}_\lambda(g)$ for small τ . It remains to prove that $\phi(Z)$ is the general solution of (37,40) for $\lambda = 0$. From (13) we have

$$Z'' = Z(gg') = B'' D''^{-1} = (AB' + BD')(CB' + DD')^{-1}, \quad (44)$$

which is not affected by C' and gives $Z'' = Z$ for dilations and Lorentz transformations ($B' = 0$) ■

Remark 4.2. In the last theorem, we are implicitly restricting ourselves to gauge transformations $g' \in S(U(2)^2)$, which means $\det(g') = \det(U_1 U_2) = 1$. If we allow for transformations $g' \in U(2)^2$ with $\det(g') \neq 1$ (like $e^{i\alpha} I$) and we want them to leave physical wave functions strictly invariant $\psi(gg') = \psi(g)$ (i.e., we restrict ourselves to representations with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$), we must choose a symmetrical form for the ground state

$$\mathcal{W}_\lambda(g) = \det(A^\dagger)^{-\lambda/2} \det(D)^{-\lambda/2} = \det(\sigma^0 - Z^\dagger Z)^{\lambda/2} \det(U_1^\dagger)^{-\lambda/2} \det(U_2)^{-\lambda/2}, \quad (45)$$

which reduces to (42) for $\det(U_1 U_2) = 1$.

Moreover, instead of (40), we could have chosen the complementary constraint $P_\mu^L \psi = 0$ which would have led us to a anti-holomorphic (“acceleration”) representation $\psi_\lambda(g) = \check{\mathcal{W}}_\lambda(g) \phi(Z^\dagger)$ with the new ground state

$$\check{\mathcal{W}}_\lambda(g) = \det(A)^{-\lambda/2} \det(D^\dagger)^{-\lambda/2} = \det(\sigma^0 - Z^\dagger Z)^{\lambda/2} \det(U_1)^{-\lambda/2} \det(U_2^\dagger)^{-\lambda/2}, \quad (46)$$

which, for $g \in SU(2, 2)$, reduces to:

$$\check{\mathcal{W}}_\lambda(g) = \det(A)^{-\lambda} = \det(\sigma^0 - Z Z^\dagger)^{\lambda/2} \det(U_1)^{-\lambda} = \overline{\mathcal{W}_\lambda(g)}.$$

Therefore, the BRP-like symmetry $K_\mu^L \leftrightarrow P_\mu^L, D^L \rightarrow -D^L$ in (7) manifest here as a charge conjugation and time reversal (CT) operations. See later on Section 6 for more details on a “BRP-CPT connection” proposal inside the conformal group. □

4.1.2 Irreducible representation, Haar measure and Bergman kernel

The finite left-action of G on physical wave functions (41),

$$\begin{aligned} [\mathcal{U}_g^L \psi_\lambda](g) &= \psi_\lambda(g'^{-1}g) = \det(D(g'^{-1}g))^{-\lambda} \phi(Z') \\ &= \mathcal{W}_\lambda(g) \det(D^\dagger - B'^\dagger Z)^{-\lambda} \phi(Z'), \\ Z' &\equiv Z(g'^{-1}g) = (A'^\dagger Z - C'^\dagger)(D^\dagger - B'^\dagger Z)^{-1}, \end{aligned} \quad (47)$$

provides a unitary irreducible representation of G under the invariant scalar product

$$\langle \psi_\lambda | \psi'_\lambda \rangle = \int_G d\mu^L(g) \overline{\psi_\lambda(g)} \psi'_\lambda(g) \quad (48)$$

given through the left-invariant Haar measure [the exterior product of left-invariant one-forms (25)] which can be decomposed as:

$$\begin{aligned} d\mu^L(g) &= c \bigwedge_{\alpha,\beta=1}^4 \vartheta_\beta^\alpha = c \det(\vartheta_{\beta\nu}^{\alpha\mu}) \bigwedge_{\mu,\nu=1}^4 dx_\mu^\nu \\ &= c d\mu(g)^L|_{G/H} d\mu^L(g)|_H, \\ d\mu^L(g)|_{G/H} &= \det(\sigma^0 - ZZ^\dagger)^{-4} |dZ|, \\ d\mu(g)|_H &= dv(U_1) dv(U_2), \end{aligned} \quad (49)$$

where we are denoting by $dv(U)$ the Haar measure on $U(2)$, which can be in turn decomposed as:

$$\begin{aligned} dv(U) &\equiv dv(U)|_{U(2)/U(1)^2} dv(U)|_{U(1)^2}, \\ dv(U)|_{U(2)/U(1)^2} &= dv(U)|_{\mathbb{S}^2} \equiv ds(U) = (1 + z\bar{z})^{-2} |dz|, \\ dv(U)|_{U(1)^2} &\equiv d\alpha d\beta. \end{aligned} \quad (50)$$

We have used the Iwasawa decomposition of an element g given in (20,21) and denoted by $|dz|$ and $|dZ|$ the Lebesgue measures in \mathbb{C} and \mathbb{C}^4 , respectively. The normalization constant

$$c = \pi^{-4} (\lambda - 1)(\lambda - 2)^2 (\lambda - 3) \left(\frac{(2\pi)^3}{2} \right)^{-2} \quad (51)$$

is fixed so that the ground state (42) is normalized, i.e. $\langle \mathcal{W}_\lambda | \mathcal{W}_\lambda \rangle = 1$ (see Appendix B of Ref. [31] for orthogonality properties), the factor $(2\pi)^3/2$ actually being the volume $v(U(2))$. The scalar product (48) is finite as long as $\lambda \geq 4$.

The infinitesimal generators of (47) are the right-invariant vector fields $R_\beta^\alpha(g)$ in (26) and constitute the operators (observables) of our quantum theory. For example, from the general expression (47), we can compute the finite left-action of dilations $g' = e^{i\tau D}$ ($B' = 0 = C'$ and $A' = e^{-i\tau/2} \sigma^0 = D'^\dagger$) on physical wave functions,

$$\psi_\lambda(g'g) = e^{i\lambda\tau} \mathcal{W}_\lambda(g) \phi(e^{i\tau} Z),$$

or infinitesimally:

$$\begin{aligned} D^R \psi_\lambda(g) &= -\frac{1}{2}(R_1^1 + R_2^2 - R_3^3 - R_4^4) \psi_\lambda(g) = \mathcal{W}_\lambda(g) \left(\lambda + \sum_{i,j=1}^2 Z_{ij} \frac{\partial}{\partial Z_{ij}} \right) \phi(Z) \\ &\equiv \mathcal{W}_\lambda(g) D_\lambda \phi(Z), \end{aligned} \quad (52)$$

where we have defined the restriction of the dilation operator on holomorphic functions as:

$$D_\lambda \equiv \lambda + \sum_{i,j=1}^2 Z_{ij} \frac{\partial}{\partial Z_{ij}}, \quad (53)$$

for future use. As we justified in Remark 3.3, the dilation generator D^R plays the role of the Hamiltonian operator of this theory

$$\hat{\mathcal{H}} = -i \frac{\partial}{\partial \tau} = D^R. \quad (54)$$

The conformal or mass dimension λ can be then interpreted as the zero point (vacuum) energy and the corresponding eigenfunctions are homogeneous polynomials $\phi_n(Z)$ of a certain degree (eigenvalue) n , according to Euler's theorem. We shall come back to this question later in Theorem 4.3.

Let us introduce bracket notation and write:

$$\mathcal{W}_\lambda(g) \equiv \langle g | \lambda, 0 \rangle = \langle \lambda, 0 | \mathcal{U}_{g^{-1}}^L | \lambda, 0 \rangle, \quad \psi_\lambda(g) \equiv \langle g | \psi_\lambda \rangle. \quad (55)$$

Here we are implicitly making use of the Coherent-States machinery (see e.g. [51, 52]). Actually, we are denoting by $|g\rangle \equiv \mathcal{U}_g^L | \lambda, 0 \rangle$ the set of vectors in the orbit of the ground (“fiducial”) state $| \lambda, 0 \rangle$ (the lowest-weight vector) under the left action of the group G (this set is called a family of *covariant coherent states* in the literature [51, 52]). We can easily calculate the coherent state overlap:

$$\begin{aligned} \langle g' | g \rangle &= \langle \lambda, 0 | \mathcal{U}_{g'^{-1}g}^L | \lambda, 0 \rangle = \mathcal{W}_\lambda(g^{-1}g') = \det(D(g^{-1}g'))^{-\lambda} = \det(D^\dagger D' - B^\dagger B')^{-\lambda} \\ &= \det(D^\dagger)^{-\lambda} \det(\sigma^0 - (BD^{-1})^\dagger B' D'^{-1})^{-\lambda} \det(D')^{-\lambda} \\ &= \overline{\mathcal{W}_\lambda(g)} \det(\sigma^0 - Z^\dagger Z')^{-\lambda} \mathcal{W}_\lambda(g'). \end{aligned} \quad (56)$$

The set of coherent states $\{|g\rangle, g \in G\}$ constitutes a tight frame (see [31] for a proof in the context of Conformal Wavelets) with resolution of unity:

$$1 = \int_G d\mu(g) |g\rangle \langle g|.$$

Actually, the coherent state overlap (56) is a reproducing kernel satisfying the integral equation of a projector operator

$$\langle g | g'' \rangle = \int_G d\mu^L(g') \langle g | g' \rangle \langle g' | g'' \rangle$$

and the propagator equation

$$\psi_\lambda(g') = \int_G d\mu^L(g) \langle g' | g \rangle \psi_\lambda(g).$$

Since the ground state \mathcal{W}_λ is a fixed common factor of all the wave functions (41), we could factor it out and define the restricted left-action

$$[\mathcal{U}_{g'}^\lambda \phi](Z) \equiv \mathcal{W}_\lambda^{-1}(g) [\mathcal{U}_{g'}^L \psi_\lambda](g) = \det(D'^\dagger - B'^\dagger Z)^{-\lambda} \phi(Z') \equiv \phi'(Z) \quad (57)$$

of G on the arbitrary (holomorphic) part ϕ of ψ_λ , instead of (47). In standard (induced) representation theory, the factor $\det(D'^\dagger - B'^\dagger Z)^{-\lambda}$ is called a “multiplier” (Radon-Nicodým derivative) and fulfils cocycle properties. For the representation (57) of G on holomorphic functions $\phi(Z)$ to be unitary, the left- G -invariant Haar measure (49) has to be accordingly modified as:

$$d\mu_\lambda(Z, Z^\dagger) \equiv c_\lambda |\mathcal{W}_\lambda(g)|^2 d\mu^L(g)|_{G/H} = c_\lambda \det(\sigma^0 - ZZ^\dagger)^{\lambda-4} |dZ|, \quad (58)$$

where $d\mu^L(g)|_{G/H}$ in (49) is the projection of the left- G -invariant Haar measure $d\mu^L(g)$ onto G/H . Roughly speaking, we are integrating out the coordinates of H and redefining the normalization constant c in (51) as $c_\lambda = c/v(U(2)) = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)$ so that the unit constant function $\phi(Z) = 1$ (the ground state) is normalized (see [31] for orthogonality properties). As before, we could also introduce a modified bracket notation $\phi(Z) \equiv (Z | \phi)$ and a new set $\{|Z\rangle, Z \in \mathbb{D}\}$ of coherent states in the Hilbert space $\mathcal{H}_\lambda(\mathbb{D}) = L^2(\mathbb{D}, d\mu_\lambda)$ of analytic square-integrable holomorphic functions ϕ on \mathbb{D} . The new coherent state overlap $(Z | Z')$ is nothing but the so called reproducing Bergman’s kernel $K_\lambda(Z, Z')$. It is related to (56) by:

$$K_\lambda(Z', Z) = (Z' | Z) = \frac{\langle g' | g \rangle}{\mathcal{W}_\lambda(g') \overline{\mathcal{W}_\lambda(g)}} = \det(\sigma^0 - Z'^\dagger Z)^{-\lambda}. \quad (59)$$

We notice that, unlike $|g\rangle$, the coherent state $|Z\rangle$ is not normalized. In fact,

$$\mathcal{K}_\lambda(Z, Z^\dagger) \equiv \ln(Z | Z) \quad (60)$$

is nothing but the Kähler potential, which defines \mathbb{D} as a Kähler manifold with local complex coordinates $Z = z_\mu \sigma^\mu$, an Hermitian Riemannian metric g and a corresponding closed two-form ω

$$ds^2 = g^{\mu\nu} dz_\mu \odot d\bar{z}_\nu, \quad \omega = -ig^{\mu\nu} dz_\mu \wedge d\bar{z}_\nu, \quad g^{\mu\nu} \equiv \frac{\partial^2 \mathcal{K}_\lambda}{\partial z_\mu \partial \bar{z}_\nu}, \quad (61)$$

where \odot denotes symmetrization. We shall come back to the Riemannian structure of \mathbb{D} and \mathbb{T} and the connection with the BRP later on Section 5.

4.1.3 Schwinger's theorem, orthonormal basis and closure relations

As already commented after Eq. (52), we are interested in calculating an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D})$ made of Hamiltonian eigenfunctions $\varphi_J(Z) \equiv (Z \mid \lambda, J)$, where J denotes a set of indices. This orthonormal basis would provide us with a new resolution of the identity

$$1 = \sum_J |\lambda, J\rangle \langle \lambda, J|.$$

Actually, we shall identify $\varphi_J(Z)$ by looking at the expansion of the Bergman's kernel

$$K_\lambda(Z', Z) = (Z' \mid Z) = \sum_J (Z' \mid \lambda, J) (\lambda, J \mid Z) = \sum_J \varphi_J(Z') \overline{\varphi_J(Z)}.$$

Thus, the Bergman's kernel plays here the role of a generating function. To be more precise:

Theorem 4.3. *The infinite set of polynomials*

$$\varphi_{q_1, q_2}^{j, m}(Z) = \sqrt{\frac{2j+1}{\lambda-1} \binom{m+\lambda-2}{\lambda-2} \binom{m+2j+\lambda-1}{\lambda-2}} \det(Z)^m \mathcal{D}_{q_1, q_2}^j(Z), \quad (62)$$

with

$$\begin{aligned} \mathcal{D}_{q_1, q_2}^j(Z) = & \sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}} \sum_{p=\max(0, q_1+q_2)}^{\min(j+q_1, j+q_2)} \binom{j+q_2}{p} \binom{j-q_2}{p-q_1-q_2} \\ & \times z_{11}^p z_{12}^{j+q_1-p} z_{21}^{j+q_2-p} z_{22}^{p-q_1-q_2} \end{aligned} \quad (63)$$

the standard Wigner's \mathcal{D} -matrices (j is a non-negative half-integer), verifies the following closure relation (the reproducing Bergman kernel):

$$\sum_{j \in \mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1, q_2=-j}^j \overline{\varphi_{q_1, q_2}^{j, m}(Z)} \varphi_{q_1, q_2}^{j, m}(Z') = \frac{1}{\det(\sigma^0 - Z^\dagger Z')^\lambda} \quad (64)$$

and constitute an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D})$.

This theorem has been proven in [31]. It turns out to be rooted in a extension of the Schwinger's formula:

Theorem 4.4. (Schwinger's Master Theorem) *The identity*

$$\sum_{j \in \mathbb{N}/2} t^{2j} \sum_{q=-j}^j \mathcal{D}_{qq}^j(X) = \frac{1}{\det(\sigma^0 - tX)} \quad (65)$$

holds for any 2×2 matrix X , with t an arbitrary parameter.

The abovementioned extension of the Theorem 4.4 can be stated as:

Theorem 4.5. (λ -Extended Schwinger's Master Theorem) *For every $\lambda \in \mathbb{N}, \lambda \geq 2$ and every 2×2 complex matrix X the following identity holds:*

$$\sum_{j \in \mathbb{N}/2} \frac{2j+1}{\lambda-1} \sum_{m=0}^{\infty} t^{2j+2m} \binom{m+\lambda-2}{\lambda-2} \binom{m+2j+\lambda-1}{\lambda-2} \det(X)^m \sum_{q=-j}^j \mathcal{D}_{qq}^j(X) = \det(\sigma^0 - tX)^{-\lambda}. \quad (66)$$

We address the interested reader to Ref. [31] for a complete proof.

Sketch of proof of Theorem 4.3: Assuming the validity of (66) and replacing $tX = Z^\dagger Z'$ in it, we have:

$$\sum_{j \in \mathbb{N}/2} \frac{2j+1}{\lambda-1} \sum_{m=0}^{\infty} \binom{m+\lambda-2}{\lambda-2} \binom{m+2j+\lambda-1}{\lambda-2} \det(Z^\dagger Z')^m \sum_{q=-j}^j \mathcal{D}_{qq}^j(Z^\dagger Z') = \frac{1}{\det(\sigma^0 - Z^\dagger Z')^\lambda}. \quad (67)$$

Using determinant and Wigner's \mathcal{D} -matrix rules

$$\det(Z^\dagger Z')^n \sum_{q=-j}^j \mathcal{D}_{qq}^j(Z^\dagger Z') = \det(Z^\dagger)^n \det(Z')^n \sum_{q_2, q_1=-j}^j \overline{\mathcal{D}_{q_1 q_2}^j(Z)} \mathcal{D}_{q_1 q_2}^j(Z')$$

and the definition of the functions (62), we see that (67) reproduces (64). On the other hand, the number of linearly independent polynomials $\prod_{i,j=1}^2 z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^2 n_{ij}$ is $(n+1)(n+2)(n+3)/6$, which coincides with the number of linearly independent polynomials (62) with degree of homogeneity $n = 2m + 2j$. This proves that the set of polynomials (62) is a basis for analytic functions $\phi \in \mathcal{H}_\lambda(\mathbb{D}_4)$. Moreover, this basis turns out to be orthonormal under the projected integration measure (58). We address the interested reader to the Appendix B of Ref. [31] for a proof. ■

Remark 4.6. The set (62) constitutes a basis of Hamiltonian eigenfunctions with energy eigenvalues E_n^λ (the homogeneity degree) given by:

$$\hat{\mathcal{H}}_\lambda \varphi_{q_1, q_2}^{j, m} = E_n^\lambda \varphi_{q_1, q_2}^{j, m}, \quad E_n^\lambda = \lambda + n, \quad n = 2j + 2m, \quad (68)$$

with $\hat{\mathcal{H}}_\lambda = D_\lambda$ defined in (53). Each energy level E_n^λ is then $(n+1)(n+2)(n+3)/6$ times degenerated. The spectrum is equi-spaced and bounded from below, with $E_0^\lambda = \lambda$ playing the role of a zero-point energy. At this stage it is interesting to compare our Hamiltonian choice with others in the literature like [53] studying a $SU(2, 2)$ -harmonic oscillator on the phase space \mathbb{D} . In this case the quantum Hamiltonian is chosen to be the Toeplitz operator corresponding to the square of the distance with respect to the $SU(2, 2)$ -invariant Kähler metric (61) on the phase space \mathbb{D} . □

4.2 Conformal spinning quantum particles

Let us use the following notation for

$$X_0 = \sum_{\alpha=1}^4 \lambda_\alpha X_\alpha^\alpha = \lambda D + s_1 \Sigma_1^{(3)} + s_2 \Sigma_2^{(3)} + \kappa I, \quad (69)$$

where

$$\Sigma_1^{(3)} = X_1^1 - X_2^2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_2^{(3)} = X_3^3 - X_4^4 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

stand for the third spin components and I the 4×4 identity matrix. The identification (69) implies that the spin labels of the representation of the subgroup $SU(2)^2$ are $s_1 \equiv (\lambda_1 - \lambda_2)/2$ and $s_2 \equiv (\lambda_3 - \lambda_4)/2$. The conformal dimension is $\lambda = (\lambda_3 + \lambda_4 - \lambda_1 - \lambda_2)/2$ and $\kappa = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/4$ is the (trace) $U(1)$ quantum number. We shall choose, without loss of generality, $\kappa = 0$, which means that λ remains integer (as in the spin-less case) and that we are restricting ourselves to representations of $SU(2, 2) \subset U(2, 2)$.

Theorem 4.7. *The general solution to (35) and (36) can be factorized as:*

$$\psi_\lambda^{s_1, s_2}(g) = \mathcal{W}_\lambda^{s_1, s_2}(g) \phi(Z, z_1, z_2), \quad (70)$$

where the ground state

$$\begin{aligned} \mathcal{W}_\lambda^{s_1, s_2}(g) &= \det(A^\dagger)^{-\lambda_s/2} \det(D)^{-\lambda_s/2} \mathcal{D}_{s_1, s_1}^{s_1}(U_1^\dagger) \mathcal{D}_{-s_2, -s_2}^{s_2}(U_2) \\ &= \det(A^\dagger)^{-\lambda_s/2} \det(D)^{-\lambda_s/2} \bar{a}_1^{2s_1} \bar{d}_2^{2s_2} \\ &= \det(\Delta_1 U_1^\dagger)^{-\lambda_s/2} \det(\Delta_2 U_2)^{-\lambda_s/2} (\delta_1 e^{-i\alpha_1})^{2s_1} (\delta_2 e^{i\beta_2})^{2s_2} \\ &= \det(\sigma^0 - Z^\dagger Z)^{\frac{\lambda_s}{2}} (1 + \bar{z}_1 z_1)^{-s_1} (1 + \bar{z}_2 z_2)^{-s_2} \\ &\quad \times e^{-i\alpha_1(2s_1 - \lambda_s/2)} e^{i\beta_1 \lambda_s/2} e^{-i\alpha_2 \lambda_s/2} e^{i\beta_2(2s_2 - \lambda_s/2)}, \end{aligned} \quad (71)$$

with $\lambda_s \equiv \lambda - s_1 - s_2$, is a particular solution of (35, 36) and ϕ is the general solution for the trivial representation $\lambda_\alpha = 0$ of $G^0 = U(1)^4$ (actually, an arbitrary, analytic holomorphic function of Z, z_1, z_2), for the decomposition (20, 21) of an element $g \in G$.

Proof: On the one hand, from (43) we conclude that the factors $\det(D)^{-\lambda}$ and $\mathcal{D}_{-s_2, -s_2}^{s_2}(U_2)$, with $U_2 = (DD^\dagger)^{-1/2} D$ fulfill the holomorphicity conditions (36) for $(\beta, \alpha) = (1, 3), (2, 3), (1, 4), (2, 4)$. Moreover, $U_1^\dagger = A^\dagger (AA^\dagger)^{-1/2}$ and we have that

$$A''^\dagger = A(gg')^\dagger = A'^\dagger A^\dagger + C'^\dagger B^\dagger = A'^\dagger (A^\dagger + (C' A'^{-1})^\dagger B^\dagger) = A'^\dagger (A^\dagger + Z' B^\dagger)$$

is not affected by $Z'^\dagger = C' A'^{-1}$ either, according to the definition (13). On the other hand, for $g' \in H$ we have that

$$\begin{aligned} \bar{a}'' &= \bar{a}(gg') = \bar{a}\bar{a}' + \bar{b}\bar{c}' = \bar{a}'(\bar{a} - z'\bar{b}) \\ d'' &= d(gg') = cb' + dd' = d'(d + z'c) \end{aligned}$$

are not affected by $\bar{z}' = -c'/a' = \bar{b}'/\bar{d}'$, according to the definition (21). This proves that the ground state (71) fulfills the holomorphicity conditions (36) for $(\beta, \alpha) = (1, 2), (3, 4)$.

It remains to prove the gauge conditions (35) or their finite counterpart (34) for $g_0 \in G^0 = U(1)^4$. Finite right (gauge) dilations $g_0 = e^{i\tau D}$ leave $\mathcal{W}_\lambda^{s_1, s_2}(gg_0) = e^{i\lambda\tau} \mathcal{W}_\lambda^{s_1, s_2}(g)$ invariant up to the phase $\mathcal{U}_0^\lambda(g_0) = e^{i\lambda\tau}$ (a character of G^0), where we have used that $\det(\cdot)$ and $\mathcal{D}^s(\cdot)$ are homogeneous of degree 2 and $2s$, respectively. Infinitesimally, it means that $D^L \psi_\lambda^{s_1, s_2} = \lambda \psi_\lambda^{s_1, s_2}$. For $g_0^{(1,2)} = e^{i\alpha \Sigma_{1,2}^3}$ the ground state transforms as expected:

$$\mathcal{W}_\lambda^{s_1, s_2}(gg_0^{(1,2)}) = \mathcal{U}_0^\lambda(g_0^{(1,2)}) \mathcal{W}_\lambda^{s_1, s_2}(g), \quad \mathcal{U}_0^\lambda(g_0^{(1,2)}) = e^{2is_{1,2}\alpha}.$$

Infinitesimally, it means that

$$\Sigma_{1,2}^{L(3)} \mathcal{W}_\lambda^{s_1, s_2} = 2s_{1,2} \mathcal{W}_\lambda^{s_1, s_2}, \quad \begin{cases} \Sigma_1^{L(3)} \equiv L_1^1 - L_2^2, \\ \Sigma_2^{L(3)} \equiv L_3^3 - L_4^4. \end{cases} \quad (72)$$

Moreover, one can easily check that $\mathcal{W}_\lambda^{s_1, s_2}(gg_0) = \mathcal{W}_\lambda^{s_1, s_2}(g)$ for diagonal $U(1)$ transformations $g_0 = e^{i\theta} I$, that is, $\kappa = 0$. Finally, using similar arguments to those employed in (44), we can assert that $z'_{1,2} = z_{1,2}(gg_0) = z_{1,2}$, $\forall g_0 \in G^0$, which ends up proving the gauge conditions (34) ■

Remark 4.8. Instead of (36), we could have chosen the complementary constraint $L_\alpha^\beta \psi = 0$, $\forall \alpha < \beta$ which would have led us to a anti-holomorphic representation. □

As in Eq. (47), we can compute the finite left-action of G on physical wave functions (70). In particular, for the case of dilations $g' = e^{i\tau' D}$ (i.e., $B' = 0 = C'$ and $A' = e^{-i\tau'/2} \sigma^0 = D'^\dagger$) we have:

$$\psi_\lambda^{s_1, s_2}(g'^{-1}g) = e^{i\lambda\tau'} \mathcal{W}_\lambda^{s_1, s_2}(g) \phi(e^{i\tau'} Z, z_1, z_2),$$

or infinitesimally:

$$D^R \psi_\lambda^{s_1, s_2}(g) = \mathcal{W}_\lambda^{s_1, s_2}(g) \left(\lambda + \sum_{i,j=1}^2 Z_{ij} \frac{\partial}{\partial Z_{ij}} \right) \phi(Z, z_1, z_2). \quad (73)$$

Comparing this expression with (52), we realize that the spin coordinates z_1, z_2 do not contribute to the degree of homogeneity of ϕ under dilations, as they correspond to “internal” (versus space-time-momentum) degrees of freedom.

As in the previous subsection, we can introduce a modified bracket notation $\phi(Z, z_1, z_2) \equiv (Z, z_1, z_2 \mid \phi)$ and a set $\{|Z, z_1, z_2\rangle, Z \in \mathbb{D}, z_1, z_2 \in \mathbb{C}\}$ of coherent states in the Hilbert space $\mathcal{H}_\lambda^{s_1, s_2}(\mathbb{F})$ of analytic measurable holomorphic functions ϕ on the twelve-dimensional pseudo-flag manifold $\mathbb{F} = U(2, 2)/U(1)^4$, locally $\mathbb{D} \times \overline{\mathbb{C}}^2$, with integration measure

$$d\mu_\lambda^{s_1, s_2}(Z, z_1, z_2; Z^\dagger, \bar{z}_1, \bar{z}_2) \equiv d\mu_{\lambda_s}(Z, Z^\dagger) \frac{2s_1 + 1}{\pi} ds(U_1) \frac{2s_2 + 1}{\pi} ds(U_2), \quad (74)$$

where $d\mu_{\lambda_s}(Z, Z^\dagger)$ and $ds(U)$ are defined in (58) and (50), respectively. Note that the square-integrability condition $\lambda \geq 4$ in $\mathcal{H}_\lambda(\mathbb{D})$ becomes $\lambda_s \geq 4$ in $\mathcal{H}_\lambda^{s_1, s_2}(\mathbb{F})$. The constant factor $(2s_1 + 1)/\pi$ is introduced so that the following set of functions is normalized.

Theorem 4.9. *The infinite set of polynomials*

$$\begin{aligned}\check{\varphi}_{j,q_1,q_2}^{m,m_1,m_2}(Z, z_1, z_2) &\equiv (-1)^{m_1+s_1} \varphi_{q_1,q_2}^{j,m}(Z) \frac{\mathcal{D}_{s_1,-m_1}^{s_1}(U_1^\dagger) \mathcal{D}_{m_2,-s_2}^{s_2}(U_2)}{\mathcal{D}_{s_1,s_1}^{s_1}(U_1^\dagger) \mathcal{D}_{-s_2,-s_2}^{s_2}(U_2)} \\ &= \varphi_{q_1,q_2}^{j,m}(Z) \sqrt{\binom{2s_1}{m_1+s_1} \binom{2s_2}{m_2+s_2}} z_1^{m_1+s_1} z_2^{m_2+s_2},\end{aligned}\quad (75)$$

(with $\varphi_{q_1,q_2}^{j,m}$ in (62) replacing $\lambda \rightarrow \lambda_s$) provides an orthonormal basis of $\mathcal{H}_\lambda^{s_1,s_2}(\mathbb{F})$. The closure relation:

$$\begin{aligned}\sum_{j \in \mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1,q_2=-j}^j \sum_{m_1=-s_1}^{s_1} \sum_{m_2=-s_2}^{s_2} \check{\varphi}_{j,q_1,q_2}^{m,m_1,m_2}(Z', z'_1, z'_2) \overline{\check{\varphi}_{j,q_1,q_2}^{m,m_1,m_2}(Z, z_1, z_2)} \\ = (Z', z'_1, z'_2 \mid Z, z_1, z_2)\end{aligned}\quad (76)$$

gives the reproducing Bergman's kernel for spinning particles:

$$\begin{aligned}K_\lambda^{s_1,s_2}(Z', z'_1, z'_2; Z, z_1, z_2) &= (Z', z'_1, z'_2 \mid Z, z_1, z_2) \\ &= \det(\sigma^0 - Z^\dagger Z')^{-\lambda_s} (1 + \bar{z}_1 z'_1)^{2s_1} (1 + \bar{z}_2 z'_2)^{2s_2}.\end{aligned}\quad (77)$$

Proof: Assuming the orthonormality of (62) (see Appendix B of Ref. [31]), and realizing that

$$\int_{\mathbb{C}} \sqrt{\binom{2s}{m}} \bar{z}^m \sqrt{\binom{2s}{m'}} z^{m'} \frac{2s+1}{\pi} ds(U) = \delta_{m,m'}, \quad m, m' = 0, \dots, 2s,$$

we prove the orthonormality of the functions (75). Moreover, the number of linearly independent polynomials $\prod_{i,j=1}^2 z_{ij}^{n_{ij}} \prod_{i=1}^2 z_i^{n_i}$ with $0 \leq n_i \leq 2s_i$ and fixed $n = \sum_{i,j=1}^2 n_{ij}$ is $(2s_1+1)(2s_2+1)(n+1)(n+2)(n+3)/6$, which coincides with the number of linearly independent polynomials (75) with degree of homogeneity $n = 2m+2j$ in the coordinates Z . This proves that the set of polynomials (75) is a basis for analytic functions $\mathcal{H}_\lambda^{s_1,s_2}(\mathbb{F})$.

It just remains to prove the closure relation (76). This proof reduces to that of Theorem 4.3 when noting the binomial identity $\sum_{m=0}^{2s} \binom{2s}{m} (\bar{z}z')^m = (1 + \bar{z}z')^{2s}$ or the Wigner \mathcal{D} -matrix property

$$\sum_{n=-s}^s \mathcal{D}_{sn}^s(U) \mathcal{D}_{ns}^s(U') = \mathcal{D}_{ss}^s(UU'). \blacksquare$$

Remark 4.10. At this point it is interesting to compare our construction with others in the literature like [8], where the proposed basis functions

$$\Phi_{j,q_1,q_2}^{m,m_1,m_2}(A, D, Z) = \mathcal{D}_{j_1,m_1}^{j_1}(A^T) \mathcal{D}_{m_2,j_2}^{j_2}(D) \varphi_{q_1,q_2}^{j,m}(Z) \quad (78)$$

do not form an orthogonal set unless a coupling between orbital angular momentum j with spin j_1, j_2 by means of Clebsch-Gordan coefficients is made:

$$\tilde{\Phi}_{j,j_1,j_2}^{m,p_1,p_2} = \sum_{m_1,m_2,q_1,q_2} C(j, q_1; s_1, m_1 - s_1 | j_1, p_1) C(j, q_2; s_2, m_2 - s_2 | j_2, p_2) \Phi_{j,q_1,q_2}^{m,m_1,m_2}.$$

Moreover, the fact that $U_1 = (AA^\dagger)^{-1/2}A$ and $U_2 = (DD^\dagger)^{-1/2}D$ introduces a new contribution of $\mathcal{D}^{j_1}(A)$ and $\mathcal{D}^{j_2}(D)$ to the integration measure $d\mu_\lambda^{j_1,j_2}$, with respect to $\mathcal{D}^{j_1}(U_1)$ and $\mathcal{D}^{j_2}(U_2)$, such that the square-integrability condition becomes $\lambda \geq 4 + 2j_1 + 2j_2$. \square

The Hamiltonian of our spinning particle is $\hat{\mathcal{H}} = -i\frac{\partial}{\partial\tau}$ with τ given by (33). Its expression in terms of right-invariant vector fields R_β^α is then

$$\hat{\mathcal{H}} = \sum_{\alpha=1}^4 \frac{1}{\lambda_\alpha} R_\alpha^\alpha = \rho_0 D^R + \rho_1 \Sigma_1^{R(3)} + \rho_2 \Sigma_2^{R(3)} + \rho_3 I, \quad (79)$$

with:

$$\rho_0 = \frac{4\lambda(\lambda^2 - s_1^2 - s_2^2)}{(\lambda^2 - 4s_1^2)(\lambda^2 - 4s_2^2)}, \quad \rho_1 = \frac{-4s_1}{\lambda^2 - 4s_1^2}, \quad \rho_2 = \frac{-4s_2}{\lambda^2 - 4s_2^2}, \quad \rho_3 = \frac{4\lambda(s_2^2 - s_1^2)}{(\lambda^2 - 4s_1^2)(\lambda^2 - 4s_2^2)},$$

and $\Sigma_{1,2}^{R(3)}$ the right-invariant version of (72). In order to compare with the spin-less case, we can always renormalize

$$\hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}/\rho_0 = \hat{\mathcal{H}}' = D^R + \varrho_1 \Sigma_1^{R(3)} + \varrho_2 \Sigma_2^{R(3)} + \varrho_3 I, \quad (80)$$

with $\varrho_\alpha = \rho_\alpha/\rho_0$. We can interpret $\varrho_{1,2}$ as constant “magnetic fields” (oriented along the “ z ” direction) coupled to the spin degrees of freedom $\Sigma_{1,2}^{R(3)}$. The set (75) constitutes a basis of eigenfunctions of the Hamiltonian (80) with eigenvalues (energy levels) given by:

$$E_{n,q_1,q_2}^{\lambda,m_1,m_2} = \lambda + \varrho_3 + n + \varrho_1(m_1 + q_1) + \varrho_2(m_2 + q_2), \quad n = 2j + 2m. \quad (81)$$

Comparing this energy eigenvalues with the energy spectrum (68) of the spin-less Hamiltonian $\hat{\mathcal{H}} = D^R$, we realize that the zero-point energy has been shifted from λ to $\lambda + \varrho_3 - s_1\varrho_1 - s_2\varrho_2$. Like in the (anomalous) *Zeeman effect*, the introduction of spin leads to an splitting of a spin-less spectral line E_n^λ into $(2s_1 + 1)(2s_2 + 1)$ components in the presence of a “static magnetic field” $\varrho_{1,2}$.

5 Relation with the tube domain realization

In this section we shall translate some expressions obtained from the complex Cartan domain (16) into the forward tube domain (17), where we enjoy more (Minkowskian) intuition. We shall restrict ourselves to the scalar case, since it is representative of the more general case.

5.1 Tube domain as a homogeneous space of $SU(2, 2)$

As we have already said, the forward tube domain \mathbb{T} is naturally homeomorphic to the quotient G/H in the realization of G in terms of matrices

$$f = \begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix} \quad (82)$$

which preserve $\Gamma = \gamma^0$, instead of $\Gamma = \gamma^5$; that is, $f^\dagger \gamma^0 f = \gamma^0$. Both realizations of G are related by the map (19), which can be explicitly written as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \Upsilon^{-1} f \Upsilon = \frac{1}{2} \begin{pmatrix} R + iS - iT + Q & -R + iS + iT + Q \\ -R - iS - iT + Q & R - iS + iT + Q \end{pmatrix}. \quad (83)$$

The identification of \mathbb{T} with the quotient G/H is given through

$$W(f) = i(R - iS)(Q + iT)^{-1}. \quad (84)$$

Hence, the left translation $f' \rightarrow ff'$ of G on itself induces a left action of G on \mathbb{T} given by:

$$W = W(f') \rightarrow W' = W(ff') = (RW + S)(TW + Q)^{-1}. \quad (85)$$

Setting $W = x_\mu \sigma^\mu$, and making use of the standard homomorphism (spinor map) between $SL(2, \mathbb{C})$ and $SO^+(3, 1)$ given by: $W' = RW R^\dagger \leftrightarrow x'^\mu = \Lambda_\nu^\mu x^\nu$, $R \in SL(2, \mathbb{C})$, $\Lambda_\nu^\mu \in SO^+(3, 1)$, the transformations (4) can be recovered from (85) as follows:

- i) Standard Lorentz transformations, $x'^\mu = \Lambda_\nu^\mu(\omega)x^\nu$, correspond to $T = S = 0$ and $R = Q^{-1\dagger} \in SL(2, \mathbb{C})$.
- ii) Dilations correspond to $T = S = 0$ and $R = Q^{-1} = \rho^{1/2}I$
- iii) Spacetime translations equal $R = Q = \sigma^0$ and $S = b_\mu \sigma^\mu$, $T = 0$.
- iv) Special conformal transformations correspond to $R = Q = \sigma^0$ and $T = a_\mu \sigma^\mu$, $S = 0$ by noting that $\det(\sigma^0 + TW) = 1 + 2ax + a^2x^2$.

5.2 Irreducible representations, Haar measure and Bergman kernel

Let us see the expression of the wave functions (41) in the tube domain \mathbb{T} . Performing the change of variables (83) in (41) we get

$$\psi_\lambda(f) = \det(Q + iT)^{-\lambda} 2^{2\lambda} \det(\sigma^0 - iW)^{-\lambda} \phi(Z(W)) \equiv \Omega_\lambda(f) \tilde{\phi}(W), \quad (86)$$

where we have defined a new ground state Ω_λ and a new function $\tilde{\phi}$ as:

$$\Omega_\lambda(f) \equiv \det(Q + iT)^{-\lambda}, \quad \tilde{\phi}(W) \equiv 2^{2\lambda} \det(\sigma^0 - iW)^{-\lambda} \phi(Z(W)). \quad (87)$$

In the same manner, the coherent-state overlap (56) can be cast as

$$\langle f' | f \rangle = \det(Q^\dagger - iT^\dagger)^{-\lambda} \det\left(\frac{i}{2}(W^\dagger - W')\right)^{-\lambda} \det(Q' + iT')^{-\lambda} \quad (88)$$

Since the ground state Ω_λ is a fixed common factor of all the wave functions (86), we can factor it out (as we did in (57) with \mathcal{W}_λ) and define the restricted action

$$\begin{aligned} [\tilde{\mathcal{U}}_f^\lambda \tilde{\phi}](Z) &\equiv \Omega_\lambda^{-1}(f) [\mathcal{U}_{f'}^L \psi_\lambda](f) \\ &= \det(R'^\dagger - T'^\dagger W)^{-\lambda} \tilde{\phi}((Q'^\dagger W - S'^\dagger)(R'^\dagger - T'^\dagger W)^{-1}) \equiv \tilde{\phi}'(W) \end{aligned} \quad (89)$$

of G on the arbitrary (holomorphic) part $\tilde{\phi}$ of ψ_λ . The Radon-Nicodým derivative is now $\det(R'^\dagger - T'^\dagger W)^{-\lambda}$. The representation (89) of G on holomorphic functions $\tilde{\phi}(W)$ is unitary with respect to the re-scaled integration measure

$$d\tilde{\mu}_\lambda(W, W^\dagger) \equiv |\Omega_\lambda(f)|^2 d\tilde{\mu}^L(f) \big|_{G/H} = \frac{c_\lambda}{2^4} \det\left(\frac{i}{2}(W^\dagger - W)\right)^{\lambda-4} |dW|, \quad (90)$$

where we are using $|dW|$ as a shorthand for the Lebesgue measure $\bigwedge_{i,j=1}^2 d\Re w_{ij} d\Im w_{ij}$ on \mathbb{T} . To arrive at (90), firstly, we have performed the Cayley transformation (18) in the projected integration measure:

$$\begin{aligned} d\mu^L(g) \big|_{G/H} &= c_\lambda \det(\sigma^0 - ZZ^\dagger)^{-4} |dZ| \rightarrow \\ d\tilde{\mu}^L(f) \big|_{G/H} &= \frac{c_\lambda}{2^4} \det\left(\frac{i}{2}(W^\dagger - W)\right)^{-4} |dW|, \end{aligned} \quad (91)$$

taking into account that $\det(\sigma^0 - ZZ^\dagger) = \det(2i(W^\dagger - W)) |\det(\sigma^0 - iW)|^{-2}$ and the Jacobian determinant $|dZ|/|dW| = 2^{12} |\det(\sigma^0 - iW)|^{-8}$, and secondly, we have written

$$|\Omega_\lambda(f)|^2 = \det(Q^\dagger - iT^\dagger)^{-\lambda} \det(Q + iT)^{-\lambda} = \det\left(\frac{i}{2}(W^\dagger - W)\right)^\lambda$$

by making use of (84) and its hermitian conjugate.

As in (59), we could also introduce a modified bracket notation $\tilde{\phi}(W) \equiv (W|\tilde{\phi})$ and a new set $\{|W\rangle, W \in \mathbb{T}\}$ of coherent states in the Hilbert space $\mathcal{H}_\lambda(\mathbb{T})$ of analytic measurable holomorphic functions φ on \mathbb{T} . The new coherent state overlap $(W|W')$ is the new Bergman's kernel $\tilde{K}_\lambda(W', W)$. It is related to (88) by:

$$\tilde{K}_\lambda(W', W) = (W' | W) = \frac{\langle f' | f \rangle}{\Omega_\lambda(f') \overline{\Omega}_\lambda(f)} = \det\left(\frac{i}{2}(W^\dagger - W')\right)^{-\lambda}. \quad (92)$$

We again notice that, unlike $|f\rangle$, the coherent state $|W\rangle$ is not normalized. Now, the Kähler potential is $\ln(W|W)$, which defines \mathbb{T} as a Kähler manifold too.

The identification (87) actually provides an isometry between the spaces of analytic holomorphic functions $\mathcal{H}_\lambda(\mathbb{D})$ and $\mathcal{H}_\lambda(\mathbb{T})$. Let us formally state it.

Proposition 5.1. *The correspondence*

$$\begin{aligned} \mathcal{S}_\lambda : \mathcal{H}_\lambda(\mathbb{D}) &\longrightarrow \mathcal{H}_\lambda(\mathbb{T}) \\ \phi &\longmapsto \mathcal{S}_\lambda \phi \equiv \tilde{\phi}, \end{aligned}$$

with

$$\tilde{\phi}(W) = 2^{2\lambda} \det(I - iW)^{-\lambda} \phi(Z(W)) \quad (93)$$

and $Z(W)$ given by the Cayley transformation(18), is an isometry, that is:

$$\langle \phi | \phi' \rangle_{\mathcal{H}_\lambda(\mathbb{D})} = \langle \mathcal{S}_\lambda \phi | \mathcal{S}_\lambda \phi' \rangle_{\mathcal{H}_\lambda(\mathbb{T})}. \quad (94)$$

Moreover, \mathcal{S}_λ is an intertwiner (equivariant map) of the representations (47) and (89), that is:

$$\mathcal{U}_\lambda = \mathcal{S}_\lambda^{-1} \tilde{\mathcal{U}}_\lambda \mathcal{S}_\lambda. \quad (95)$$

Proof: The isometry property is proven by construction from (87). The intertwining relation (95) can be explicitly written as:

$$\begin{aligned} [\mathcal{U}_\lambda \phi](Z) &= \det(D^\dagger - B^\dagger Z)^{-\lambda} \phi((A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1}) = \\ \left[\mathcal{S}_\lambda^{-1} \tilde{\mathcal{U}}_\lambda \tilde{\phi} \right](Z) &= \det(I - iW)^\lambda \det(R^\dagger - T^\dagger W)^{-\lambda} \det(I - iW')^{-\lambda} \phi(Z(W')), \end{aligned} \quad (96)$$

where $W' = (Q^\dagger W - S^\dagger)(R^\dagger - T^\dagger W)^{-1}$. On the one hand, we have that the argument of ϕ is:

$$\begin{aligned} Z(W') &= (I + iW')(I - iW')^{-1} \\ &= ((R^\dagger - T^\dagger W) + i(Q^\dagger W - S^\dagger)) ((R^\dagger - T^\dagger W) - i(Q^\dagger W - S^\dagger))^{-1} \\ &= ((R^\dagger - iS^\dagger) + i(Q^\dagger + iT^\dagger)W) ((R^\dagger + iS^\dagger) - i(Q^\dagger - iT^\dagger)W)^{-1}. \end{aligned}$$

Taking now into account the map (83) we have:

$$\begin{aligned} Z(W') &= ((A^\dagger - C^\dagger) + i(A^\dagger + C^\dagger)W) ((D^\dagger - B^\dagger) - i(D^\dagger + B^\dagger)W)^{-1} \\ &= (A^\dagger(I + iW) - C^\dagger(I - iW)) (D^\dagger(I - iW) - B^\dagger(I + iW))^{-1} \\ &= (A^\dagger Z - C^\dagger) (D^\dagger - B^\dagger Z)^{-1}, \end{aligned}$$

as desired. On the other hand, we have that

$$\begin{aligned} (I - iW')(R^\dagger - T^\dagger W) &= (R^\dagger - T^\dagger W) - i(Q^\dagger W - S^\dagger) = (R^\dagger + iS^\dagger) - i(Q^\dagger - iT^\dagger)W \\ &= (D^\dagger - B^\dagger) - i(D^\dagger + B^\dagger)W = D^\dagger(I - iW) - B^\dagger(I + iW) = (D^\dagger - B^\dagger Z)(I - iW) \end{aligned}$$

which implies

$$\det(I - iW)^\lambda \det(R^\dagger - T^\dagger W)^{-\lambda} \det(I - iW')^{-\lambda} = \det(D^\dagger - B^\dagger Z)^{-\lambda}$$

That is, the equality of multipliers in (96). ■

As a direct consequence of Proposition 5.1, the set of functions defined by

$$\tilde{\varphi}_{q_1, q_2}^{j, m}(W) \equiv 2^{2\lambda} \det(I - iW)^{-\lambda} \varphi_{q_1, q_2}^{j, m}(Z(W)), \quad (97)$$

with $\varphi_{q_1, q_2}^{j, m}$ defined in (62), constitutes an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{T})$ and the closure relation

$$\sum_{j \in \mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q, q'=-j}^j \overline{\tilde{\varphi}_{q', q}^{j, m}(W)} \tilde{\varphi}_{q', q}^{j, n}(W') = \det\left(\frac{i}{2}(W^\dagger - W')\right)^{-\lambda}, \quad (98)$$

renders again the reproducing Bergman kernel (92).

5.3 Kähler structures, Born's reciprocity and maximal acceleration

As we said for the Cartan domain \mathbb{D} in (60) and (61), the Kähler potential

$$\tilde{\mathcal{K}}_\lambda(W, W^\dagger) \equiv \ln(W | W) = -\ln |\Omega_\lambda(f)|^2 = -\lambda \ln(\Im(w))^2 = -\lambda \ln y^2 \quad (99)$$

defines \mathbb{T} as a Kähler manifold with local complex coordinates $W = w_\mu \sigma^\mu$, $w_\mu = x_\mu + iy_\mu$, an Hermitian Riemannian metric

$$g^{\mu\nu} \equiv \frac{\partial^2 \tilde{\mathcal{K}}_\lambda}{\partial w_\mu \partial \bar{w}_\nu} = -\frac{\lambda}{2y^2} \left(\eta^{\mu\nu} - 2 \frac{y_\mu y_\nu}{y^2} \right). \quad (100)$$

and a corresponding closed two-form ω

$$\omega = -ig^{\mu\nu} dw_\mu \wedge d\bar{w}_\nu. \quad (101)$$

The line element

$$ds^2 = g^{\mu\nu} dw_\mu d\bar{w}_\nu = -\frac{\lambda}{2y^2} \left(\eta^{\mu\nu} - 2 \frac{y_\mu y_\nu}{y^2} \right) (dx_\mu dx_\nu + dy_\mu dy_\nu) \quad (102)$$

turns out to be positive and provides a conformal counterpart of the Born's line element (2). The two-form (101) defines the Poisson bracket:

$$\{a, b\} \equiv ig_{\mu\nu} \left(\frac{\partial a}{\partial w_\mu} \frac{\partial b}{\partial \bar{w}_\nu} - \frac{\partial b}{\partial w_\mu} \frac{\partial a}{\partial \bar{w}_\nu} \right) \quad (103)$$

for the inverse metric

$$g_{\mu\nu} = -\frac{2}{\lambda} \left(\eta^{\mu\nu} y^2 - 2y_\mu y_\nu \right). \quad (104)$$

so that $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$. In particular, we have that:

$$\{x_\mu, y_\nu\} = -\frac{1}{2} g_{\mu\nu},$$

which differs from $\{x_\mu, y_\nu\} = \eta_{\mu\nu}$; that is, x_μ and y_ν are not “canonical” coordinates. However, we can define a proper conjugate four-momentum $p_\mu \equiv \lambda y_\mu / y^2$ which gives the desired (canonical) Poisson bracket

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu}, \quad (105)$$

as can be checked by direct computation. The line element (102) then becomes:

$$ds^2 = -\frac{1}{2\lambda} (\eta^{\mu\nu} p^2 - 2p^\mu p^\nu) (dx_\mu dx_\nu + \frac{\lambda^2}{p^4} dp_\mu dp_\nu). \quad (106)$$

Note the close resemblance between the coordinates, $dx^\mu(K^\nu) = -2x^\mu x^\nu + x^2 \eta^{\mu\nu}$, of the vector field K^ν in (5) and the metric coefficients $(-2p^\mu p^\nu + p^2 \eta^{\mu\nu})$ in (106) under the interchange $x_\mu \leftrightarrow p_\mu$. The line element (106) of the (curved) manifold \mathbb{T} is the conformal counterpart of the Born’s line element (2) in the (flat) complex Minkowski space $\mathbb{C}^{1,3}$, both of them considered as phase spaces of relativistic (conformal) particles. Concerning the extension of BRP to the case of curved spacetimes, see also [54] for the construction a reciprocal general relativity theory as a local gauge theory of the quaplectic group of [21, 22].

Remember that one could deduce the existence of a maximal acceleration from the positivity of the Born’s line element (3). The existence of a maximal acceleration inside the conformal group does not seem to be apparent from (106), although there are other arguments supporting the existence of a bound a_{\max} for proper accelerations. One of them was given time ago in Ref. [55], where the authors analyzed the physical interpretation of the singularities, $1 + 2ax + a^2 x^2 = 0$, of the conformal transformations to a uniformly accelerating frame [last transformation in (4)]. When applying the transformation to an extended object of size ℓ , an upper-limit to the proper acceleration, $a_{\max} \simeq c^2/\ell$, is shown to be necessary in order that the tenets of special relativity not be violated (see [55] for more details).

In a coming paper [27], we shall provide an alternative proof of the existence of a maximal acceleration inside the conformal group. It is related to the Unruh effect (vacuum radiation in uniformly accelerated frames) and turns out to be a consequence of the finiteness of the radiated energy (black body spectrum). Contrary to other approaches to the Unruh effect, a bound for the proper acceleration does not necessarily imply a bound for the temperature.

6 Comments and outlook

We have revised the use of complex Minkowski 8-dimensional space (more precisely, the domains \mathbb{D} and \mathbb{T}) as a base for the construction of conformal-invariant quantum (field) theory, either as a phase space or a configuration space [the last case related to Lagrangians of type (29)]. We have followed a gauge-invariant Lagrangian approach (of nonlinear sigma-model type) and we have used a generalized Dirac method for the quantization of constrained systems, which resembles in some aspects the particular approach to quantizing coadjoint orbits of a group G developed in, for instance, [9].

One could think of these 8-dimensional domains as the replacement of space-time at short distances or high momentum transfers, as it is implicit in the original BRP [15, 16], the standard relativity theory being then the limit $\ell_{\min} \rightarrow 0$. Group-theoretical revisions of the BRP, replacing the Poincaré by the Canonical (or Quaplectic) group of reciprocal relativity, have been proposed in [21, 22]. In this article we put a (conformal) BRP-like forward, as a natural symmetry inside the conformal group $SO(4, 2)$ and the replacement of space-time by the 8-dimensional conformal domain \mathbb{D} or \mathbb{T} at short distances. Actually, we feel tempted to establish a connection between *holomorphicity* \leftrightarrow *chirality* and BRP \leftrightarrow CPT symmetry inside the conformal group. Indeed, the definition of P_μ and K_μ in (10) is linked to the right- and left-handed projectors $(1+\gamma^5)/2$ and $(1-\gamma^5)/2$, respectively. According to the (conformal) BRP-like symmetry (7), conformal physics is symmetric under the interchange $P_\mu \leftrightarrow K_\mu$, as long as we perform a proper-time reversal $D \rightarrow -D$. On the other hand, $P_\mu \leftrightarrow K_\mu$ entails a swapping of chirality $(1+\gamma^5)/2 \leftrightarrow (1-\gamma^5)/2$, a complex conjugation $\psi_\lambda(g) \leftrightarrow \check{\psi}_\lambda(g) = \overline{\psi_\lambda(g)}$ (remember the discussion in Remark 4.2) and a parity inversion $\sigma_\mu \leftrightarrow \check{\sigma}_\mu = \sigma^\mu$. Nevertheless, at this stage, a BRP \leftrightarrow CPT connection inside the conformal group is just conjectural and it is still premature to draw any physical conclusions based on it. It is not either the main objective of this paper.

In this article we have considered a particular class of representations (discrete series) of the conformal group, although other possibilities could also be tackled. For example, we could consider the new (vector and pseudo-vector) combinations

$$\tilde{P}_\mu \equiv \frac{1}{2}(P_\mu + K_\mu), \quad \tilde{K}_\mu \equiv \frac{1}{2}(P_\mu - K_\mu),$$

with new commutation relations:

$$[\tilde{P}_\mu, \tilde{K}_\nu] = \eta_{\mu\nu}D, \quad [\tilde{P}_\mu, \tilde{P}_\nu] = M_{\mu\nu}, \quad [\tilde{K}_\mu, \tilde{K}_\nu] = -M_{\mu\nu}. \quad (107)$$

Unlike in formulas (37) and (40), the fact that now $[D, \tilde{K}_\mu] = -\tilde{P}_\mu$ precludes the imposition of $D^L, M_{\mu\nu}^L$ and \tilde{K}_μ^L as a compatible set of constraints on wave functions. Instead, we could impose

$$M_{\mu\nu}^L \psi = 0, \quad \tilde{K}_\mu^L \psi = 0$$

together with the Casimir (8) constraint $C_2^L \psi = m_{00}^2 \psi$, which leads to

$$((D^L)^2 + (\tilde{P}^L)^2) \psi = m_{00}^2 \psi,$$

This equation could be seen as a *generalized* Klein-Gordon equation ($P^2 \psi = m_0^2 \psi$), with D replacing P_0 as the (proper) time generator and m_{00} replacing the Poincaré-invariant mass m_0 , as a “conformally-invariant mass” (see e.g.[56] for the formulation of other conformally-invariant massive field equations of motion in generalized Minkowski space). This means that Cauchy hypersurfaces have dimension 4. In other words, the Poincaré time is a dynamical variable, on an equal footing with position, the usual Poincaré Hamiltonian P_0 suffering Heisenberg indeterminacy relations too. Instead of the proper time

(dilation) generator D , one could also consider the new combination $\tilde{P}_0 = (P_0 + K_0)/2$ as the new Hamiltonian of our theory (see [57] for this choice).

In a non-commutative geometry setting [58], the non-vanishing commutators (107), or those of the position operators X_μ in (39) giving spin generators [48, 49], can be seen as a sign of the granularity (non-commutativity) of space-time in conformal-invariant theories, along with the existence of a minimal length or, equivalently, a maximal acceleration.

The appearance of a maximal acceleration inside the conformal group will be manifest in analyzing the Unruh effect from a group-theoretical perspective [27]. In a previous paper [28], vacuum radiation in uniformly accelerated frames was related to a spontaneous breakdown of the conformal symmetry. In fact, in conformally-invariant quantum field theory, one can find degenerated pseudo-vacua (which turn out to be coherent states of conformal zero-modes) which are stable (invariant) under Poincaré transformations but are excited under accelerations and lead to a black-body spectrum. The same spontaneous-symmetry-breaking mechanism applies to general $U(N, M)$ -invariant quantum field theories, where an interesting connection between “curvature and statistics” has emerged [59]. We hope this is just one of many interesting physical phenomena that remain to be unravelled inside conformal-invariant quantum field theory.

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